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DEFORMATION QUANTIZATION: QUANTUM MECHANICS LIVES AND WORKS IN PHASE-SPACE

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Wigner's quasi-probability distribution function in phase-space is a special (Weyl) representation of the density matrix. It has been useful in describing quantum transport in quantum optics; nuclear physics; decoherence (eg, quantum computing); quantum chaos; "Welcher Weg" discussions; semiclassical limits. It is also of importance in signal processing. Nevertheless, a remarkable aspect of its internal logic, pioneered by Moyal, has only emerged in the last quarter-century: It furnishes a third, alternative, formulation of Quantum Mechanics, independent of the conventional Hilbert Space, or Path Integral formulations. In this logically complete and self-standing formulation, one need not choose sides—coordinate or momentum space. It works in full phase-space, accommodating the uncertainty principle. This is an introductory overview of the formulation with simple illustrations.

1. Introduction

There are three logically autonomous alternative paths to quantization. The first is the standard one utilizing operators in Hilbert space, developed by Heisenberg, Schrödinger, Dirac, and others in the Twenties of the past century. The second one relies on Path Integrals, and was conceived by Dirac¹ and constructed by Feynman.

The last one (the bronze medal!) is the Phase-Space formulation, based on Wigner's (1932) quasi-distribution function² and Weyl's (1927) correspondence³ between quantum-mechanical operators and ordinary c-number phase-space functions. The crucial composition structure of these functions, which relies on the \star -product, was fully understood by Groenewold (1946)⁴, who, together with Moyal (1949)⁵, pulled the entire formulation together. Still, insights on interpretation and a full appreciation of its conceptual autonomy took some time to mature with the work of Takabayasi⁶, Baker⁷, Fairlie⁸, and others. (A brief guide to some landmark papers is provided in the Appendix.)

This complete formulation is based on the Wigner Function (WF), which is a quasi-probability distribution function in phase-space,

$$f(x, p) = \frac{1}{2\pi} \int dy \, \psi^*(x - \frac{\hbar}{2}y) e^{-iyp} \psi(x + \frac{\hbar}{2}y). \quad (1)$$

It is a special representation of the density matrix (in the Weyl correspondence, as detailed in Section 8). Alternatively, it is a generating function for all spatial autocorrelation functions of a given quantum-mechanical wavefunction $\psi(x)$.

There are several outstanding reviews on the subject ^{9,10,11}. Nevertheless, the central conceit of this review is that the above input wavefunctions may ultimately be forfeited, since the WFs are determined, in principle, as the solutions of suitable (celebrated) functional equations. Connections to the Hilbert space operator formulation of quantum mechanics may thus be ignored, in principle—even though they are provided in Section 8 for pedagogy and confirmation of the formulations’ equivalence. One might then envision an imaginary planet on which this formulation of Quantum Mechanics preceded the conventional formulation, and its own techniques and methods arose independently, perhaps out of generalizations of classical mechanics and statistical mechanics.

It is not only wavefunctions that are missing in this formulation. Beyond an all-important (noncommutative, associative, pseudodifferential) operation, the \star -product, which encodes the entire quantum mechanical action, there are no operators. Observables and transition amplitudes are phase-space integrals of c-number functions (which compose through the \star -product), weighted by the WF, as in statistical mechanics. Consequently, even though the WF is not positive-semidefinite (it can, and usually does go negative in parts of phase-space), the computation of observables and the associated concepts are evocative of classical probability theory.

This formulation of Quantum Mechanics is useful in describing *quantum* transport processes in phase space, of importance in quantum optics, nuclear physics, condensed matter, and the study of semiclassical limits of mesoscopic systems ¹² and the transition to classical statistical mechanics¹³. It is the natural language to study quantum chaos and decoherence¹⁴ (of utility in, eg, quantum computing), and provides intuition in QM interference such as in “Welcher Weg” problems¹⁵, probability flows as negative probability backflows¹⁶, and measurements of atomic systems¹⁷. The intriguing mathematical structure of the formulation is of relevance to Lie Algebras¹⁸, and has recently been retrofitted into M-theory advances linked to noncommutative geometry ¹⁹, and matrix models²⁰, which apply spacetime uncertainty principles ²¹ reliant on the \star -product. (Transverse spatial dimensions act formally as momenta, and, analogously to quantum mechanics, their uncertainty is increased or decreased inversely to the uncertainty of a given direction.)

As a significant aside, the WF has extensive practical applications in signal processing and engineering (time-frequency analysis), since time and energy (frequency) constitute a pair of Fourier-conjugate variables just like the \mathfrak{x} and \mathfrak{p} of phase

space^a

For simplicity, the formulation will be mostly illustrated for one coordinate and its conjugate momentum, but generalization to arbitrary sized phase spaces is straightforward²³, including infinite-dimensional ones, namely scalar field theory^{24,25,26,27}.

2. The Wigner Function

As already indicated, the quasi-probability measure in phase space is the WF,

$$f(x, p) = \frac{1}{2\pi} \int dy \, \psi^*(x - \frac{\hbar}{2}y) e^{-iyp} \psi(x + \frac{\hbar}{2}y). \quad (2)$$

It is obviously normalized, $\int dp dx f(x, p) = 1$. In the classical limit, $\hbar \rightarrow 0$, it would reduce to the probability density in coordinate space x , usually highly localized, multiplied by δ -functions in momentum: the classical limit is “spiky”! This expression has more $x - p$ symmetry than is apparent, as Fourier transformation to momentum-space wavefunctions yields a completely symmetric expression with the roles of x and p reversed, and, upon rescaling of the arguments x and p , a symmetric classical limit.

It is also manifestly Real^b. It is also constrained by the Schwarz Inequality to be bounded, $-\frac{2}{\hbar} \leq f(x, p) \leq \frac{2}{\hbar}$. Again, this bound disappears in the spiky classical limit.

p - or x -projection leads to marginal (bona-fide) probability densities: a spacelike shadow $\int dp f(x, p) = \rho(x)$, or else a momentum-space shadow $\int dx f(x, p) = \sigma(p)$, respectively. Either is a (bona-fide) probability density, being positive semidefinite. But neither can be conditioned on the other, as the uncertainty principle is fighting back: The WF $f(x, p)$ itself can, and most often does go negative in some areas of phase-space^{2,9}, as is illustrated below, a hallmark of QM interference in this language. (In fact, the only pure state WF which is non-negative is the Gaussian²⁸.)

The counter-intuitive “negative probability” aspects of this quasi-probability distribution have been explored and interpreted^{29,30,16}, and negative probability flows amount to legitimate probability backflows in interesting settings¹⁶. Nevertheless, the WF for atomic systems can still be measured in the laboratory, albeit indirectly¹⁷.

Smoothing f by a filter of size larger than \hbar (eg, convolving with a phase-space Gaussian) results in a positive-semidefinite function, ie it may be thought to have been coarsened to a classical distribution^{31c}.

^aThus, time varying signals are best represented in a WF as time varying spectrograms, analogously to a music score, ie the changing distribution of frequencies is monitored in time²²: even though the description is constrained and redundant, it gives an intuitive picture of the signal that a mere time profile or frequency spectrogram fails to convey. Applications abound in acoustics, speech analysis, seismic data analysis, internal combustion engine-knocking analysis, failing helicopter component vibrations, etc.

^bIn one space dimension, by virtue of non-degeneracy, it turns out to be p -even, but this is not a property used here.

^cThis one is called the Husimi distribution³², and sometimes information scientists examine it on

Among real functions, the WFs comprise a rather small, highly constrained, set. When is a real function $f(x, p)$ a bona-fide Wigner function of the form (2)? Evidently, when its Fourier transform “left-right” factorizes,

$$\tilde{f}(x, y) = \int dp e^{ipy} f(x, p) = g_L^*(x - \hbar y/2) g_R(x + \hbar y/2), \quad (3)$$

ie,

$$\frac{\partial^2 \ln \tilde{f}}{\partial(x - \hbar y/2) \partial(x + \hbar y/2)} = 0, \quad (4)$$

so $g_L = g_R$ from reality.

Nevertheless, as indicated, the WF *is* a distribution function: it provides the integration measure in phase space to yield expectation values from phase-space c-number functions. Such functions are often classical quantities, but, in general, are associated to suitably ordered operators through Weyl’s correspondence rule³. Given an operator ordered in this prescription,

$$\mathfrak{G}(\mathfrak{x}, \mathfrak{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp g(x, p) \exp(i\tau(\mathfrak{p} - p) + i\sigma(\mathfrak{x} - x)), \quad (5)$$

the corresponding phase-space function $g(x, p)$ (the “classical kernel of the operator”) is obtained by

$$\mathfrak{p} \mapsto p, \quad \mathfrak{x} \mapsto x. \quad (6)$$

That operator’s expectation value is then a “phase-space average”^{4,5},

$$\langle \mathfrak{G} \rangle = \int dx dp f(x, p) g(x, p). \quad (7)$$

The classical kernel $g(x, p)$ is often the unmodified classical expression, such as a conventional hamiltonian, $H = \frac{p^2}{2m} + V(x)$, ie the transition from classical mechanics is straightforward ; but it contains \hbar when there are quantum-mechanical ordering ambiguities, such as for the classical kernel of the square of the angular momentum $\mathfrak{L} \cdot \mathfrak{L}$: this one contains $O(\hbar^2)$ terms introduced by the Weyl ordering, beyond the classical expression L^2 . In such cases, the classical kernels may still be produced without direct consideration of operators. A more detailed discussion of the correspondence is provided in Section 8.

In this sense, expectation values of the physical observables specified by classical kernels $g(x, p)$ are computed through integration with the WF, in close analogy to classical probability theory, but for the non-positive-definiteness of the distribution

account of its non-negative feature. Nevertheless, it comes with a heavy price, as it needs to be “dressed” back to the WF for all practical purposes when expectation values are computed with it, ie it does not serve as an immediate quasi-probability distribution with no other measure³². The negative feature of the WF is, in the last analysis, an asset, and not a liability, and provides an efficient description of “beats”²².

function. This operation corresponds to tracing an operator with the density matrix (Sec 8).

3. Solving for the Wigner Function

Given a specification of observables, the next step is to find the relevant WF for a given hamiltonian. Can this be done without solving for the Schrödinger's wavefunctions ψ , i.e. not using Schrödinger's equation directly? The functional equations which f satisfies completely determine it.

Firstly, its dynamical evolution is specified by Moyal's equation. This is the extension of Liouville's theorem of classical mechanics, for a classical hamiltonian $H(x, p)$, namely $\partial_t f + \{f, H\} = 0$, to quantum mechanics, in this language^{2,5}:

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar}, \quad (8)$$

where the \star -product⁴ is

$$\star \equiv e^{\frac{i\hbar}{2}(\overrightarrow{\partial}_x \overrightarrow{\partial}_p - \overrightarrow{\partial}_p \overrightarrow{\partial}_x)}. \quad (9)$$

The right hand side of (8), dubbed the “Moyal Bracket” (the Weyl-correspondent of quantum commutators), is the essentially unique one-parameter (\hbar) associative deformation of the Poisson Brackets of classical mechanics³³. Expansion in \hbar around 0 reveals that it consists of the Poisson Bracket corrected by terms $O(\hbar)$. The equation (8) also evokes Heisenberg's equation of motion for operators, except H and f here are classical functions, and it is the \star -product which introduces noncommutativity. This language, thus, makes the link between quantum commutators and Poisson Brackets more transparent.

Since the \star -product involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments (“Bopp shifts”),

$$f(x, p) \star g(x, p) = f\left(x + \frac{i\hbar}{2} \overrightarrow{\partial}_p, p - \frac{i\hbar}{2} \overrightarrow{\partial}_x\right) g(x, p). \quad (10)$$

The equivalent Fourier representation of the \star -product is⁷:

$$\begin{aligned} f \star g &= \frac{1}{\hbar^2 \pi^2} \int dp' dp'' dx' dx'' f(x', p') g(x'', p'') \\ &= \exp\left(\frac{-2i}{\hbar} (p(x' - x'') + p'(x'' - x) + p''(x - x'))\right). \end{aligned} \quad (11)$$

An alternate integral representation of this product is³⁴

$$f \star g = (\hbar\pi)^{-2} \int dudvdwdz f(x+u, p+v) g(x+w, p+z) \exp\left(\frac{2i}{\hbar} (uz - vw)\right), \quad (12)$$

which readily displays noncommutativity and associativity.

\star -multiplication of c-number phase-space functions is in complete isomorphism to Hilbert-space operator multiplication⁴,

$$\mathfrak{A}\mathfrak{B} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp (a \star b) \exp(i\tau(\mathfrak{p} - p) + i\sigma(\mathfrak{x} - x)). \quad (13)$$

The cyclic phase-space trace is directly seen in the representation (12) to reduce to a plain product, if there is only one \star involved,

$$\int dp dx f \star g = \int dp dx fg = \int dp dx g \star f. \quad (14)$$

Moyal's equation is necessary, but does not suffice to specify the WF for a system. In the conventional formulation of quantum mechanics, systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. Time-independent pure-state Wigner functions \star -commute with H , but clearly not every function \star -commuting with H can be a bona-fide WF (eg, any \star -function of H will \star -commute with H).

Static WFs obey more powerful functional \star -genvalue equations⁸ (also see³⁵),

$$\begin{aligned} H(x, p) \star f(x, p) &= H\left(x + \frac{i\hbar}{2} \overrightarrow{\partial}_p, p - \frac{i\hbar}{2} \overrightarrow{\partial}_x\right) f(x, p) \\ &= f(x, p) \star H(x, p) = E f(x, p), \end{aligned} \quad (15)$$

where E is the energy eigenvalue of $\mathfrak{H}\psi = E\psi$. These amount to a complete characterization of the WFs³⁶.

Lemma: For real functions $f(x, p)$, the Wigner form (2) for pure static eigenstates is equivalent to compliance with the \star -genvalue equations (15) (\Re and \Im parts).

Proof:

$$\begin{aligned} &H(x, p) \star f(x, p) = \\ &= \frac{1}{2\pi} \left((p - i\frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x) \right) \int dy e^{-iy(p + i\frac{\hbar}{2} \overrightarrow{\partial}_x)} \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\ &= \frac{1}{2\pi} \int dy \left((p - i\frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x + \frac{\hbar}{2}y) \right) e^{-iy p} \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\ &= \frac{1}{2\pi} \int dy e^{-iy p} \left((i\overrightarrow{\partial}_y + i\frac{\hbar}{2} \overrightarrow{\partial}_x)^2 / 2m + V(x + \frac{\hbar}{2}y) \right) \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\ &= \frac{1}{2\pi} \int dy e^{-iy p} \psi^*(x - \frac{\hbar}{2}y) E \psi(x + \frac{\hbar}{2}y) \\ &= E f(x, p). \end{aligned} \quad (16)$$

Action of the effective differential operators on ψ^* turns out to be null.

Symmetrically,

$$\begin{aligned}
f \star H &= \\
&= \frac{1}{2\pi} \int dy e^{-iyp} \left(-\frac{1}{2m} (\vec{\partial}_y - \frac{\hbar}{2} \vec{\partial}_x)^2 + V(x - \frac{\hbar}{2}y) \right) \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y) \\
&= E f(x, p),
\end{aligned} \tag{17}$$

where the action on ψ is now trivial. Conversely, the pair of \star -eigenvalue equations dictate, for $f(x, p) = \int dy e^{-iyp} \tilde{f}(x, y)$,

$$\int dy e^{-iyp} \left(-\frac{1}{2m} (\vec{\partial}_y \pm \frac{\hbar}{2} \vec{\partial}_x)^2 + V(x \pm \frac{\hbar}{2}y) - E \right) \tilde{f}(x, y) = 0. \tag{18}$$

Hence, Real solutions of (15) must be of the form $f = \int dy e^{-iyp} \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y)/2\pi$, such that $\mathfrak{H}\psi = E\psi$. \square

The eqs (15) lead to spectral properties for WFs^{8,36}, as in the Hilbert space formulation. For instance, projective orthogonality of the \star -genfunctions follows from associativity, which allows evaluation in two alternate groupings:

$$f \star H \star g = E_f f \star g = E_g f \star g. \tag{19}$$

Thus, for $E_g \neq E_f$, it is necessary that

$$f \star g = 0. \tag{20}$$

Moreover, precluding degeneracy (which can be treated separately), choosing $f = g$ above yields,

$$f \star H \star f = E_f f \star f = H \star f \star f, \tag{21}$$

and hence $f \star f$ must be the stargenfunction in question,

$$f \star f \propto f. \tag{22}$$

Pure state f s then \star -project onto their space. In general, it can be shown^{6,36} that

$$f_a \star f_b = \frac{1}{h} \delta_{a,b} f_a. \tag{23}$$

The normalization matters⁶: despite linearity of the equations, it prevents superposition of solutions. (Quantum mechanical interference works differently here, in comportance with density matrix formalism.)

By virtue of (14), for different \star -genfunctions, the above dictates that

$$\int dp dx f g = 0. \tag{24}$$

Consequently, unless there is zero overlap for all such WFs, at least one of the two must go negative someplace to offset the positive overlap⁹—an illustration of the negative values' feature, quite far from being a liability.

Further note that integrating (15) yields the expectation of the energy,

$$\int H(x, p) f(x, p) dx dp = E \int f dx dp = E. \quad (25)$$

NB Likewise,^d integrating the above projective condition yields

$$\int f^2 dx dp = \frac{1}{h}, \quad (26)$$

that is the overlap increases to a divergent result in the classical limit, as the WFs grow increasingly more spiky.

In classical (non-negative) probability distribution theory, expectation values of non-negative functions are likewise non-negative, and thus result in standard constraint inequalities for the constituent pieces of such functions, such as, e.g., moments of the variables. But it was just seen that for WFs which go negative for an arbitrary function g , $\langle |g|^2 \rangle$ need not ≥ 0 : this can be easily seen by choosing the support of g to lie mostly in those regions of phase-space where the WF f is negative.

Still, such constraints are not lost for WFs. It turns out they are replaced by

$$\langle g^* \star g \rangle \geq 0, \quad (27)$$

and lead to the uncertainty principle³⁷. In Hilbert space operator formalism, this relation would correspond to the positivity of the norm. This expression is non-negative because it involves a real non-negative integrand for a pure state WF satisfying the above projective condition^e,

$$\int dp dx (g^* \star g) f = h \int dx dp (g^* \star g) (f \star f) = h \int dx dp (f \star g^*) \star (g \star f) = h \int dx dp |g \star f|^2. \quad (28)$$

To produce Heisenberg's uncertainty relation, one only need choose

$$g = a + bx + cp, \quad (29)$$

for arbitrary complex coefficients a, b, c . The resulting positive semi-definite quadratic form is then

$$a^* a + b^* b \langle x \star x \rangle + c^* c \langle p \star p \rangle + (a^* b + b^* a) \langle x \rangle + (a^* c + c^* a) \langle p \rangle + c^* b \langle p \star x \rangle + b^* c \langle x \star p \rangle \geq 0, \quad (30)$$

^dThis discussion applies to proper WFs, corresponding to pure states' density matrices. e.g., a sum of two WFs is not a pure state in general, and do not satisfy the condition (4). For such generalizations, the "impurity" is⁴ $\int dx dp (f - hf^2) \geq 0$, where the inequality is only saturated into an equality for a pure state. For instance, for $w \equiv (f_a + f_b)/2$ with $f_a \star f_b = 0$, the impurity is nonvanishing, $\int dx dp (w - hw^2) = 1/2$.

^eSimilarly, if f_1 and f_2 are pure state WFs, the transition probability between the respective states is also non-negative³⁸, manifestly by the same argument³⁷, namely, $\int dp dx f_1 f_2 = (2\pi\hbar)^2 \int dx dp |f_1 \star f_2|^2 \geq 0$.

for any a, b, c . The eigenvalues of the corresponding matrix are then non-negative, and thus so must be its determinant. Given

$$x \star x = x^2, \quad p \star p = p^2, \quad p \star x = px - i\hbar/2, \quad x \star p = px + i\hbar/2, \quad (31)$$

and the usual

$$(\Delta x)^2 \equiv \langle (x - \langle x \rangle)^2 \rangle, \quad (\Delta p)^2 \equiv \langle (p - \langle p \rangle)^2 \rangle, \quad (32)$$

this condition on the 3×3 matrix determinant amounts to

$$(\Delta x)^2 (\Delta p)^2 \geq \hbar^2/4 + \left(\langle (x - \langle x \rangle)(p - \langle p \rangle) \rangle \right)^2, \quad (33)$$

and hence

$$\Delta x \Delta p \geq \hbar/2. \quad (34)$$

The \hbar entered into the moments' constraint through the action of the \star -product. \square

4. Illustration: the Harmonic Oscillator

To illustrate the formalism on a simple prototype problem, one may look at the harmonic oscillator. In the spirit of this picture, one can, in fact, eschew solving the Schrödinger problem and plugging the wavefunctions into (2); instead, one may solve (15) directly for $H = (p^2 + x^2)/2$ (with $m = 1$, $\omega = 1$),

$$\left(\left(x + \frac{i\hbar}{2} \partial_p \right)^2 + \left(p - \frac{i\hbar}{2} \partial_x \right)^2 - 2E \right) f(x, p) = 0. \quad (35)$$

For this hamiltonian, the equation has collapsed to two simple PDEs. The first one, the Imaginary part,

$$(x \partial_p - p \partial_x) f = 0, \quad (36)$$

restricts f to depend on only one variable, the scalar in phase space, $z = 4H/\hbar = 2(x^2 + p^2)/\hbar$. Thus the second one, the Real part, is a simple ODE,

$$\left(\frac{z}{4} - z \partial_z^2 - \partial_z - \frac{E}{\hbar} \right) f(z) = 0. \quad (37)$$

Setting $f(z) = \exp(-z/2) L(z)$ yields Laguerre's equation,

$$\left(z \partial_z^2 + (1 - z) \partial_z + \frac{E}{\hbar} - \frac{1}{2} \right) L(z) = 0. \quad (38)$$

It is solved by Laguerre polynomials,

$$L_n = \frac{e^z \partial^n (e^{-z} z^n)}{n!}, \quad (39)$$

for $n = E/\hbar - 1/2 = 0, 1, 2, \dots$, so the \star -gen-Wigner-functions are ⁴

$$f_n = \frac{(-1)^n}{\pi} e^{-2H/\hbar} L_n(4H/\hbar);$$

$$L_0 = 1, \quad L_1 = 1 - 4H/\hbar, \quad L_2 = 8H^2/\hbar^2 - 8H/\hbar + 1, \dots \quad (40)$$

But for the Gaussian ground state, they all have zeros and go negative. These functions become spiky in the classical limit $\hbar \rightarrow 0$; e.g., the ground state Gaussian f_0 goes to a δ -function.

Their sum provides a resolution of the identity⁵,

$$\sum_n f_n = 1/\hbar. \quad (41)$$

(For the rest of this section, for algebraic simplicity, set $\hbar = 1$.)

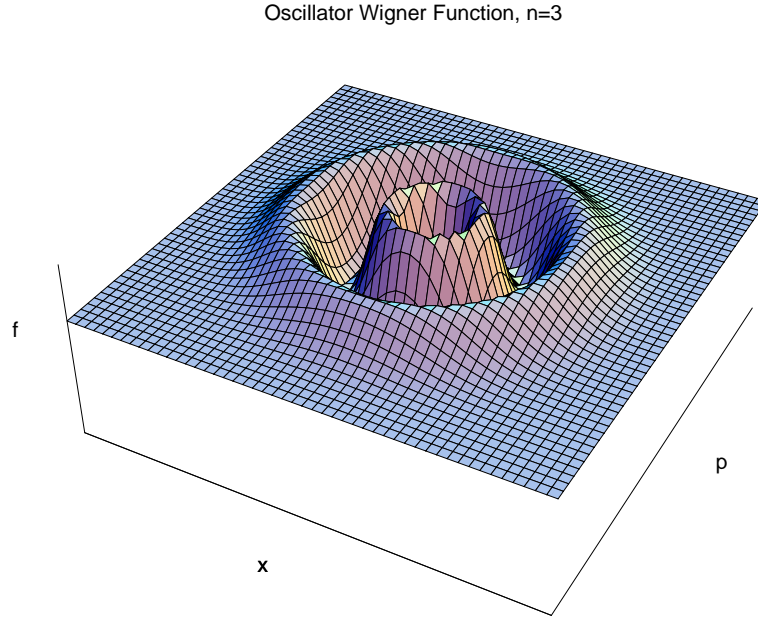


Fig. 1. WF for the 3rd excited state. Note the negative values.

Dirac's hamiltonian factorization method for the alternate algebraic solution of the same problem carries through intact, with \star -multiplication supplanting operator multiplication. That is to say,

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2}. \quad (42)$$

This motivates definition of raising and lowering functions (not operators)

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \quad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip), \quad (43)$$

where

$$a \star a^\dagger - a^\dagger \star a = 1. \quad (44)$$

The annihilation ones \star -annihilate the \star -Fock vacuum,

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2 + p^2)} = 0. \quad (45)$$

Thus, the associativity of the \star -product permits the customary ladder spectrum generation³⁶. The \star -genstates for $H \star f = f \star H$ are

$$f_n \propto (a^\dagger \star)^n f_0 (\star a)^n. \quad (46)$$

They are manifestly real, like the Gaussian ground state, and left-right symmetric; it is easy to see they are \star -orthogonal for different eigenvalues. Likewise, they can be seen by the evident algebraic normal ordering to project to themselves, since the Gaussian ground state does, $f_0 \star f_0 = f_0/h$. The corresponding coherent state WFs^{39,40} are likewise very analogous to the conventional formulation.

This type of analysis carries over well to a broader class of problems³⁶ with “essentially isospectral” pairs of partner potentials, connected with each other through Darboux transformations relying on Witten superpotentials W (cf the Pöschl-Teller potential). It closely parallels the standard differential equation structure of the recursive technique. That is, the pairs of related potentials and corresponding \star -genstate Wigner functions are constructed recursively³⁶ through ladder operations analogous to the algebraic method outlined for the oscillator.

Beyond such recursive potentials, examples of further simple systems where the \star -genvalue equations can be solved on first principles include the linear potential^{41,36,42}, the exponential interaction Liouville potentials, and their supersymmetric Morse generalizations³⁶ (also see⁴³). Further systems may be handled through the Chebyshev-polynomial numerical techniques of ref⁴⁴.

First principles phase-space solution of the Hydrogen atom is less than straightforward and complete. The reader is referred to^{11,45,46} for significant partial results.

5. Time Evolution

Moyal’s equation (8) is formally solved by virtue of associative combinatoric operations completely analogous to Hilbert space Quantum Mechanics, through definition of a \star -unitary evolution operator, a “ \star -exponential”¹¹

$$U_\star(x, p; t) = e_\star^{itH/\hbar} \equiv 1 + (it/\hbar)H(x, p) + \frac{(it/\hbar)^2}{2!}H \star H + \frac{(it/\hbar)^3}{3!}H \star H \star H + \dots, \quad (47)$$

for arbitrary hamiltonians. The solution to Moyal's equation, given the WF at $t = 0$, then, is

$$f(x, p; t) = U_{\star}^{-1}(x, p; t) \star f(x, p; 0) \star U_{\star}(x, p; t). \quad (48)$$

For the variables x and p , the evolution equations collapse to *classical* trajectories,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p, \quad (49)$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x, \quad (50)$$

where the concluding member of these two equations hold for the oscillator only. Thus, for the oscillator,

$$x(t) = x \cos t + p \sin t, \quad (51)$$

$$p(t) = p \cos t - x \sin t. \quad (52)$$

As a consequence, for the oscillator, the functional form of the Wigner function is preserved along classical phase-space trajectories⁴,

$$f(x, p; t) = f(x \cos t - p \sin t, p \cos t + x \sin t; 0). \quad (53)$$

Any oscillator WF configuration rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete quantum mechanical description^{4,47,2,25,27,48}.

Naturally, this rigid rotation in phase-space preserves areas, and thus automatically illustrates the uncertainty principle. By contrast, in general, in the conventional formulation of quantum mechanics, this result is deprived of intuitive import, or, at the very least, simplicity: upon integration in x (or p) to yield usual probability densities, the rotation induces apparent complicated shape variations of the oscillating probability density profile, such as wavepacket spreading (as evident in the shadow projections on the x and p axes of Fig 2).

Only when (as is the case for coherent states⁴⁹) a Wigner function configuration has an *additional* axial $x - p$ symmetry around its *own* center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density⁴⁸.

In Dirac's interaction representation, a more complicated interaction hamiltonian superposed on the oscillator one, leads to shape changes of the WF configurations placed on the above "turntable", and serves to generalize to scalar field theory²⁷. A more elaborate discussion of propagators is found in^{40,41}.

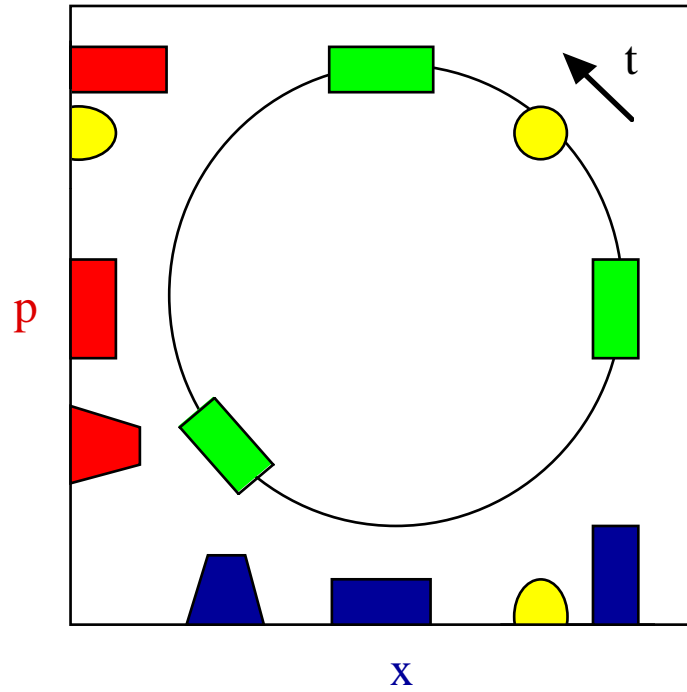


Fig. 2. Time evolution of generic WF configurations driven by an oscillator hamiltonian. The t -arrow indicates the rotation sense of x and p , and so, for fixed x and p axes, the WF shoebox configurations rotate rigidly in the opposite direction, clockwise. (The sharp angles of the WFs in the illustration are actually unphysical, and were only chosen to monitor their “spreading wavepacket” projections more conspicuously.) These x and p -projections (shadows) are meant to be intensity profiles on those axes, but are expanded on the plane to aid visualization. The circular figure represents a coherent state, which projects on either axis identically at all times, thus without shape alteration of its wavepacket through time evolution.

6. Canonical Transformations

Canonical transformations $(x, p) \mapsto (X(x, p), P(x, p))$ preserve the phase-space volume (area) element (again, take $\hbar = 1$) through a trivial Jacobian,

$$dXdP = dx dp \{X, P\} , \quad (54)$$

ie, they preserve Poisson Brackets

$$\{u, v\}_{xp} \equiv \frac{\partial u}{\partial x} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial x} , \quad (55)$$

$$\{X, P\}_{xp} = 1, \quad \{x, p\}_{XP} = 1, \quad \{x, p\} = \{X, P\}. \quad (56)$$

Upon quantization, the c-number function hamiltonian transforms “classically”, $\mathcal{H}(X, P) \equiv H(x, p)$, like a scalar. Does the \star -product remain invariant under this transformation?

Yes, for *linear* canonical transformations³⁹, but clearly *not for general canonical transformations*. Still, things can be put right, by devising general *covariant* transformation rules for the \star -product³⁶: The WF transforms in comportance with Dirac’s quantum canonical transformation theory¹.

In conventional quantum mechanics, for canonical transformations generated by $F(x, X)$,

$$p = \frac{\partial F(x, X)}{\partial x} , \quad P = -\frac{\partial F(x, X)}{\partial X} , \quad (57)$$

the energy eigenfunctions transform in a generalization of the “representation-changing” Fourier transform¹,

$$\psi_E(x) = N_E \int dX e^{iF(x, X)} \Psi_E(X) . \quad (58)$$

(In this expression, the generating function F may contain \hbar corrections to the classical one, in general—but for several simple quantum mechanical systems it manages not to⁵⁰.) Hence³⁶, there is a transformation functional for WFs, $\mathcal{T}(x, p; X, P)$, such that

$$f(x, p) = \int dXdP \mathcal{T}(x, p; X, P) \star \mathcal{F}(X, P) = \int dXdP \mathcal{T}(x, p; X, P) \mathcal{F}(X, P) , \quad (59)$$

where

$$\begin{aligned} & \mathcal{T}(x, p; X, P) \\ &= \frac{|N|^2}{2\pi} \int dY dy \exp \left(-iyp + iPY - iF^*(x - \frac{y}{2}, X - \frac{Y}{2}) + iF(x + \frac{y}{2}, X + \frac{Y}{2}) \right) . \end{aligned} \quad (60)$$

Moreover, it can be shown that³⁶,

$$H(x, p) \star \mathcal{T}(x, p; X, P) = \mathcal{T}(x, p; X, P) \star \mathcal{H}(X, P). \quad (61)$$

That is, if \mathcal{F} satisfies a \star -genvalue equation, then f satisfies a \star -genvalue equation with the same eigenvalue, and vice versa. This proves useful in constructing WFs for simple systems which can be trivialized classically through canonical transformations.

7. Omitted Miscellany

Phase-space quantization extends in several interesting directions which are not covered in such a short introduction.

Complete sets of WFs which correspond to *off-diagonal* elements of the density matrix (see next section), and which thus enable investigation of interference phenomena and the formulation of quantum mechanical perturbation theory are covered in^{47,51,40}. The spectral theory of WFs is discussed in^{11,52,40}.

Mathematical, uniqueness, and computational features of \star -products in various geometries are discussed in^{11,53,54,55,56,57}.

Alternate phase-space distributions, but always equivalent to the WF, are reviewed in^{9,58,54,32}.

8. The Weyl Correspondence

This section summarizes the bridge and equivalence of phase-space quantization to the conventional formulation of Quantum Mechanics in Hilbert space.

Weyl³ introduced an association rule mapping invertibly c-number phase-space functions $g(x, p)$ (called classical kernels) to operators \mathfrak{G} in a given ordering prescription. Specifically, $p \mapsto \mathfrak{p}$, $x \mapsto \mathfrak{x}$, and, in general,

$$\mathfrak{G}(\mathfrak{x}, \mathfrak{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp g(x, p) \exp(i\tau(\mathfrak{p} - p) + i\sigma(\mathfrak{x} - x)). \quad (62)$$

The eponymous ordering prescription requires that an arbitrary operator, regarded as a power series in \mathfrak{x} and \mathfrak{p} , be first ordered in a completely symmetrized expression in \mathfrak{x} and \mathfrak{p} , by use of Heisenberg's commutation relations, $[\mathfrak{x}, \mathfrak{p}] = i\hbar$. A term with m powers of \mathfrak{p} and n powers of \mathfrak{x} will be obtained from the coefficient of $\tau^m \sigma^n$ in the expansion of $(\tau\mathfrak{p} + \sigma\mathfrak{x})^{m+n}$. It is evident how the map yields a Weyl-ordered operator from a polynomial classical kernel. It includes every possible ordering with multiplicity one, eg,

$$6p^2x^2 \mapsto \mathfrak{p}^2\mathfrak{x}^2 + \mathfrak{x}^2\mathfrak{p}^2 + \mathfrak{p}\mathfrak{x}\mathfrak{p}\mathfrak{x} + \mathfrak{p}\mathfrak{x}^2\mathfrak{p} + \mathfrak{x}\mathfrak{p}\mathfrak{x}\mathfrak{p} + \mathfrak{x}\mathfrak{p}^2\mathfrak{x}. \quad (63)$$

In this correspondence scheme, then,

$$\text{Tr} \mathfrak{G} = \int dx dp g. \quad (64)$$

Conversely⁴, the c-number phase-space kernels $g(x, p)$ of Weyl-ordered operators $\mathfrak{G}(\mathfrak{x}, \mathfrak{p})$ are specified by $\mathfrak{p} \mapsto p$, $\mathfrak{x} \mapsto x$, or, more precisely,

$$g(x, p) = \frac{1}{(2\pi)^2} \int d\tau d\sigma e^{i(\tau p + \sigma x)} \text{Tr} \left(e^{-i(\tau\mathfrak{p} + \sigma\mathfrak{x})} \mathfrak{G} \right) =$$

$$= \frac{1}{2\pi} \int dy e^{-iyp} \langle x + \frac{\hbar}{2} y | \mathfrak{G}(\mathfrak{x}, \mathfrak{p}) | x - \frac{\hbar}{2} y \rangle, \quad (65)$$

since⁹ the above trace reduces to

$$\int dz e^{i\tau\sigma\hbar/2} \langle z | e^{-i\sigma\mathfrak{x}} e^{-i\tau\mathfrak{p}} \mathfrak{G} | z \rangle = 2\pi \int dz \langle z - \hbar\tau | \mathfrak{G} | z \rangle e^{i\sigma(\tau\hbar/2 - z)}. \quad (66)$$

Thus, the density matrix inserted in this expression⁵ yields a hermitean generalization of the Wigner function:

$$\begin{aligned} f_{mn}(x, p) &\equiv \frac{1}{2\pi} \int dy e^{-iyp} \langle x + \frac{\hbar}{2} y | \psi_n \rangle \langle \psi_m | x - \frac{\hbar}{2} y \rangle \\ &= \frac{1}{2\pi} \int dy e^{-iyp} \psi_m^*(x - \frac{\hbar}{2} y) \psi_n(x + \frac{\hbar}{2} y) = f_{nm}^*(x, p), \end{aligned} \quad (67)$$

where the $\psi_m(x)$ s are (ortho-)normalized solutions of a Schrödinger problem. (Wigner² mainly considered the diagonal elements of the density matrix (pure states), denoted above as $f_m \equiv f_{mm}$.) As a consequence, matrix elements of operators, ie, traces of them with the density matrix, are produced through mere phase-space integrals⁵,

$$\langle \psi_m | \mathfrak{G} | \psi_n \rangle = \int dx dp g(x, p) f_{mn}(x, p), \quad (68)$$

and thus expectation values follow for $m = n$, as utilized throughout in this review. Hence,

$$\langle \psi_m | \exp i(\sigma\mathfrak{x} + \tau\mathfrak{p}) | \psi_m \rangle = \int dx dp \exp i(\sigma x + \tau p) f_m(x, p), \quad (69)$$

the moment-generating functional⁵ of the Wigner distribution.

Products of Weyl-ordered operators are easily reordered into Weyl-ordered operators through the degenerate Campbell-Baker-Hausdorff identity. In a study of the uniqueness of the Schrödinger representation, von Neumann³⁴ adumbrated the composition rule of classical kernels in such operator products, appreciating that Weyl's correspondence was in fact a homomorphism. (Effectively, he arrived at a convolution representation of the star product.) Finally, Groenewold⁴ neatly worked out in detail how the classical kernels f and g of two operators \mathfrak{F} and \mathfrak{G} must compose to yield the classical kernel of $\mathfrak{F}\mathfrak{G}$,

$$\begin{aligned} \mathfrak{F}\mathfrak{G} &= \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x', p') g(x'', p'') \\ &\quad \times \exp i(\xi(\mathfrak{p} - p') + \eta(\mathfrak{x} - x')) \exp i(\xi'(\mathfrak{p} - p'') + \eta'(\mathfrak{x} - x'')) \\ &= \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x', p') g(x'', p'') \exp i((\xi + \xi')\mathfrak{p} + (\eta + \eta')\mathfrak{x}) \\ &\quad \times \exp i\left(-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right). \end{aligned} \quad (70)$$

Changing integration variables to

$$\xi' \equiv \frac{2}{\hbar}(x - x'), \quad \xi \equiv \tau - \frac{2}{\hbar}(x - x'), \quad \eta' \equiv \frac{2}{\hbar}(p' - p), \quad \eta \equiv \sigma - \frac{2}{\hbar}(p' - p), \quad (71)$$

reduces the above integral to

$$\mathfrak{F}\mathfrak{G} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \exp i(\tau(\mathfrak{p} - p) + \sigma(\mathfrak{x} - x)) (f \star g)(x, p), \quad (72)$$

where $f \star g$ is the expression (11). \square

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References

1. P Dirac, *Phys Z Sowjetunion* **3** (1933) 64-72
2. E Wigner, *Phys Rev* **40** (1932) 749-759
3. H Weyl, *Z Phys* **46** (1927) 1; reviewed in H Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931)
4. H Groenewold, *Physica* **12** (1946) 405-460
5. J Moyal, *Proc Camb Phil Soc* **45** (1949) 99-124
6. T Takabayasi, *Prog Theo Phys* **11** (1954) 341-373
7. G Baker, *Phys Rev* **109** (1958) 2198-2206
8. D Fairlie, *Proc Camb Phil Soc* **60** (1964) 581-586
9. For reviews, see
M Hillery, R O'Connell, M Scully, and E Wigner, *Phys Repts* **106** (1984) 121;
H-W Lee, *Phys Repts* **259** (1995) 147;
F Berezin, *Sov Phys Usp* **23** (1980) 763-787;
N Balasz and B Jennings, *Phys Repts* **104** (1984) 347;
R Littlejohn, *Phys Repts* **138** (1986) 193;
M Gadella, *Fortschr Phys* **43** (1995) 3, 229-264;
P Carruthers and F Zachariasen, *Rev Mod Phys* **55** (1983) 245;
A M Ozorio de Almeida, *Phys Rept* **295** (1998) 265-342;
V Tatarskii, *Sov Phys Usp* **26** (1983) 311;
L Cohen, *Time-Frequency Analysis* (Prentice Hall PTR, Englewood Cliffs, 1995);
Y Kim and M Noz, *Phase Space Picture of Quantum Mechanics*, Lecture Notes in Physics v 40 (World Scientific, Singapore, 1991);
R Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1963);
U Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, 1997)
10. M Berry, *Philos Trans R Soc London* **A287** (1977) 237-271
11. F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer, *Ann Phys* **111** (1978) 61; *ibid* 111; *Lett Math Phys* **1** (1977) 521-530

12. M Mallalieu and C Stroud, *Phys Rev* **A51** (1995) 1827-1835;
I Oppenheim and J Ross, *Phys Rev* **107** (1957) 28;
K Imre et al, *J Math Phys* **8** (1967) 1097;
Y Kim and E Wigner, *Phys Rev* **A36** (1987) 1293; *ibid* **A38** (1988) 1159
13. A Joshi and H-T Dung, *Mod Phys Lett* **B13** (1999) 143-152;
W Frensky, *Phys Rev* **B36** (1987) 1570-1578;
D Brown and P Danielewicz, *Phys Rev* **D58** (1998) 094003;
R Simon and N Mukunda, *J Opt Soc Am* **a17** (2000) 2440-2463
14. M Benedict and A Czirják, *Phys Rev* **A** (1999) 4034;
M Gell-Mann and Hartle, *Phys Rev* **D47** (1993) 3345;
W Zurek and J Paz, *Phys Rev Lett* **72** (1994) 2508;
W Zurek, *Physics Today* (October 1991) 36;
A Kolovsky, *Phys Rev Lett* **76** (1996) 340-343;
C Kiefer and E Joos, in *Quantum Future*, P Blanchard and A Jadczyk, eds (Springer, Berlin, 1999) pp 105-128 [quant-ph/9803052]
15. H Wiseman et al, *Phys Rev* **A56** (1997) 55-75
16. A Bracken and G Mello, *J Phys* **A27** (1994) 2197-2211
17. D Smithey et al, *Phys Rev Lett* **70** (1993) 1244;
T Dunn et al, *Phys Rev Lett* **74** (1995) 884;
D Leibfried et al, *Phys Rev Lett* **77** (1996) 4281;
C Kurtsiefer, T Pfau and J Mlynek, *Nature* **386** (1997) 150;
A Lvovsky et al, *Phys Rev Lett* **87** (2001) 050402
18. D Fairlie and C Zachos, *Phys Lett* **B224** (1989) 101-107;
D Fairlie, P Fletcher and C Zachos, *J Math Phys* **31** (1990) 1088-1094
19. N Seiberg and E Witten, *JHEP* **9909** (1999) 032;
for reviews, see L Castellani, *Class Quant Grav* **17** (2000) 3377-3402 [hep-th/0005210];
J Harvey, [hep-th/0102076];
M Douglas and N Nekrasov, [hep-th/0106048].
20. W Taylor, *Rev Mod Phys* **73** (2001) 419 [hep-th/0101126].
21. R Peierls, *Z Phys* **80** (1933) 763;
T Yoneya, *Mod Phys Lett* **A4** (1989) 1587;
A Jevicki and T Yoneya, *Nucl Phys* **B535** (1998) 335 [hep-th/9805069];
N Seiberg, L Susskind and N Toumbas, *JHEP* **0006** (2000) 044 [hep-th/0005015]
22. H Bartelt, K Brenner, and A Lohmann, *Opt Commun* **32** (1980) 32-38;
W Wokurek, in *Proc ICASSP '97* (Munich, 1997), pp 1435-1438 ;
L Cohen, *Time-Frequency Analysis* (Prentice Hall PTR, Englewood Cliffs, 1995);
S Qian and D Chen, *Joint Time-Frequency Analysis* (Prentice Hall PTR, Upper Saddle River, NJ, 1996);
W Mecklenbräuker and F Hlawatsch, eds, *The Wigner Distribution* (Elsevier, Amsterdam, 1997)
23. V Dodonov and V Man'ko, *Physica* **137A** (1986) 306-316
24. J Dito, *Lett Math Phys* **20** (1990) 125-134
25. B Lesche, *Phys Rev* **D29** (1984) 2270
26. H Nachbagauer, [hep-th/9703105]
27. T Curtright and C Zachos, *J Phys* **A32** (1999) 771-779
28. R Hudson, *Rep Math Phys* **6** (1974) 249-252
29. M Bartlett, *Proc Camb Phil Soc* **41** (1945) 71-73
30. R Feynman, "Negative Probability" in *Essays in Honor of David Bohm*, B Hiley and F Peat, eds, (Routledge and Kegan Paul, London, 1987); for a popular review, see D Leibfried, T Pfau, and C Monroe, *Physics Today* (April 1998) 22-28

31. N Cartwright, *Physica* **83A** (1976) 210-213
32. K Takahashi, *Prog Theo Phys Suppl* **98** (1989) 109-156
33. J Vey, *Comment Math Helv* **50** (1975) 421-454;
W Arveson, *Comm Math Phys* **89** (1983) 77-102;
P Fletcher, *Phys Lett* **B248** (1990) 323-328;
M Bertelson, M Cahen, and S Gutt, *Class Quant Grav* **14** (1997) A93-A107
M Flato, A Lichnerowicz, and D Sternheimer, *J Math Phys* **17** (1976) 1754;
M de Wilde and P Lecomte, *Lett Math Phys* **7** (1983) 487
34. J v Neumann, *Math Ann* **104** (1931) 570-578
35. K Takahashi, *Prog Theo Phys Suppl* **98** (1989) 109-156;
J Dahl, in *Energy Storage and Redistribution*, J Hinze, ed (Plenum Press, New York, 1983) pp 557-571;
L Cohen, *Jou Math Phys* **17** (1976) 1863;
W Kundt, *Z Nat Forsch* **a22** (1967) 1333-6
36. T Curtright, D Fairlie, and C Zachos, *Phys Rev* **D58** (1998) 025002
37. T Curtright and C Zachos, [hep-th/0105226]
38. R O'Connell and E Wigner, *Phys Lett* **85A** (1981) 121-126
39. D Han, Y Kim, and M Noz, *Phys Rev* **A37** (1988) 807;
Y Kim and E Wigner, *ibid* **A38** (1988) 1159; *ibid* **A36** (1987) 1293
40. T Curtright, T Uematsu, and C Zachos, *J Math Phys* **42** (2001) 2396-2415 [hep-th/0011137]
41. G García-Calderón and M Moshinsky, *J Phys* **A13** (1980) L185
42. Go Torres-Vega, A Zúñiga-Segundo, and J Morales-Guzmán, *Phys Rev* **A53** (1996) 3792-3797
43. A Frank, A Rivera, and K Wolf, *Phys Rev* **A61** (2000) 054102
44. M Hug, C Menke, and W Schleich, *Phys Rev* **A57** (1998) 3188-3205; *ibid* 3206-3224
45. J Gracia-Bondía, *Phys Rev* **A30** (1984) 691-697
46. J Dahl and M Springborg, *Mol Phys* **47** (1982) 1001; *Phys Rev* **A59** (1999) 4099-4100
47. M Bartlett and J Moyal, *Proc Camb Phil Soc* **45** (1949) 545-553
48. C Zachos and T Curtright, *Prog Theo Phys Suppl* **135** (1999) 244-258 [hep-th/9903254]
49. M Hennings, T Smith, and D Dubin, *J Phys* **A28** (1995) 6779-6807; *ibid* 6809-6856;
J Klauder and B Skagerstam, *Coherent States* (World Scientific, Singapore, 1985)
50. E Davis and G Ghandour, [quant-ph/9905002];
T Hakioglu, [quant-ph/0011076];
J-H Kim and H-W Lee, *Can J Phys* **77** (1999) 411-425
51. L Wang and R O'Connell, *Found Phys* **18** (1988) 1023-1033
52. D Fairlie and C Manogue, *J Phys* **A24** (1991) 3807-3815
53. F Hansen, *Rept Math Phys* **19** (1984) 361-381;
C Roger and V Ovsienko, *Russ Math Surv* **47** (1992) 135-191
54. C Zachos, *J Math Phys* **41** (2000) 5129-5134, [hep-th/9912238]; C Zachos, [hep-th/0008010] in *Integrable Hierarchies and Modern Physical Theories*, H Aratyn and A Sorin, eds, NATO Science Series II 18 (Kluwer AP, Dordrecht, 2001), pp 423-435
55. M Kontsevich (1997) [q-alg/9709040]
56. F Berezin, *Com Math Phys* **40** (1975) 153-174;
B Fedosov, *J Diff Geom* **40** (1994) 213-238;
N Reshetikhin and L Takhtajan, [math.QA/9907171];
I Kishimoto, [hep-th/0103018];
57. F Antonsen, [gr-qc/9710021]
58. F Bayen, in *Group Theoretical Methods in Physics*, W Beiglböck et al, eds, *Lecture Notes in Physics* **94** (Springer-Verlag, Heidelberg, 1979) pp 260-271;

- A Voros, *Phys Rev* **A40** (1989) 6814-6825
59. J Sylvester, *Johns Hopkins University Circulars* **I** (1882) 241-242; *ibid* **II** (1883) 46; *ibid* **III** (1884) 7-9. Summarized in *The Collected Mathematics Papers of James Joseph Sylvester* (Cambridge University Press, 1909) v III
60. P A M Dirac, *Rev Mod Phys* **17** (1945) 195-199

Appendix A Selected Publications on Phase-space Quantization

Implicitly, the bulk of the formulation is contained in Groenewold's and Moyal's seminal papers. But this has been a slow story of emerging connections and chains of ever sharper reformulations, and confidence in the autonomy of the formulation arose very slowly. As a result, attribution of critical milestones cannot avoid subjectivity: it cannot automatically highlight merely the *earliest* occurrence of a construct, unless that has also been compelling enough to yield an "indefinite stay against confusion" about the logical structure of the formulation.

H Weyl (1927)³ introduces the correspondence of "Weyl-ordered" operators to phase-space (c-number) kernel functions (as well as discrete QM application of Sylvester's (1883)⁵⁹ clock and shift matrices).

J von Neumann (1931)³⁴, in a technical aside off a study of the uniqueness of Schrödinger's representation, includes an implicit version of the \star -product which promotes Weyl's correspondence rule to full isomorphism between Weyl-ordered operator multiplication and \star -convolution of kernel functions.

E Wigner (1932)² (and L Szilard, unpublished) introduces the eponymous phase-space distribution function controlling quantum mechanical diffusive flow in phase space. It specifies the time evolution of this function and applies it to quantum statistical mechanics.

H Groenewold (1946)⁴. A seminal but somewhat unappreciated paper which fully understands the Weyl correspondence and produces the WF as the classical kernel of the density matrix. It reinvents and streamlines von Neumann's construct into the standard \star -product, in a systematic exploration of the isomorphism between Weyl-ordered operator products and their kernel function compositions. It further works out the harmonic oscillator WF.

J Moyal (1949)⁵ amounts to a grand synthesis: It establishes an independent formulation of quantum mechanics in phase space. It systematically studies all expectation values of Weyl-ordered operators, and identifies the Fourier transform of their moment-generating function (their characteristic function) to the Wigner Function. It further interprets the subtlety of the "negative probability" formalism and reconciles it with the uncertainty principle. Not least, it recasts the time evolution of the Wigner Function through a deformation of the Poisson Bracket into the Moyal Bracket (the commutator of \star -products, i.e., the Weyl correspondent of the Heisenberg commutator), and thus opens up the way for a systematic study of the semiclassical limit. Before publication, Dirac has already been impressed by this work, contrasting it to his own ideas on functional integration, in Bohr's *Festschrift*⁶⁰.

T Takabayasi (1954)⁶ investigates the fundamental projective normalization condition for pure state Wigner functions, and exploits Groenewold's link to the conventional density matrix formulation. It further illuminates the diffusion of wavepackets.

G Baker (1958)⁷ envisions the logical autonomy of the formulation, based on postulating the projective normalization condition. It resolves measurement subtleties in the correspondence principle and appreciates the significance of the anticommutator of the \star -product as well, thus shifting emphasis to the \star -product itself, over and above its commutator.

D Fairlie (1964)⁸ (also see ³⁵) explores the time-independent counterpart to Moyal's evolution equation, which involves the \star -product, beyond mere Moyal Bracket equations, and derives (instead of postulating) the projective orthonormality conditions for the resulting Wigner functions. These now allow for a unique and full solution of the quantum system, in principle (without any reference to the conventional Hilbert-space formulation). Thus, autonomy of the formulation is fully recognized.

M Berry (1977)¹⁰ elucidates the subtleties of the semiclassical limit, ergodicity, integrability, and the singularity structure of Wigner function evolution.

F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer (1978)¹¹ analyzes systematically the deformation structure and the uniqueness of the formulation, with special emphasis on spectral theory, and consolidates it mathematically. It provides explicit solutions to standard problems and introduces influential technical innovations, such as the \star -exponential.

T Curtright, D Fairlie, and C Zachos (1998)³⁶ demonstrates more directly the equivalence of the time-independent \star -genvalue problem to the Hilbert space formulation, and hence its logical autonomy; formulates Darboux isospectral systems in phase space; establishes the covariant transformation rule for general *nonlinear* canonical transformations (with reliance on the classic work of P Dirac (1933)¹; and thus furnishes explicit solutions of nontrivial practical problems on first principles, without recourse to the Hilbert space formulation. Efficient techniques, e.g. for perturbation theory, are based on generating functions for complete sets of Wigner functions in T Curtright, T Uematsu, and C Zachos (2001)⁴⁰. A self-contained derivation of the uncertainty principle is provided in T Curtright and C Zachos (2001)³⁷.

M Hug, C Menke, and W Schleich (1998)⁴⁴ introduce techniques for numerical solution of \star -equations on a basis of Chebyshev polynomials.